Quantization of a Harmonic Oscillator

— Utilizing Lagrange Equation

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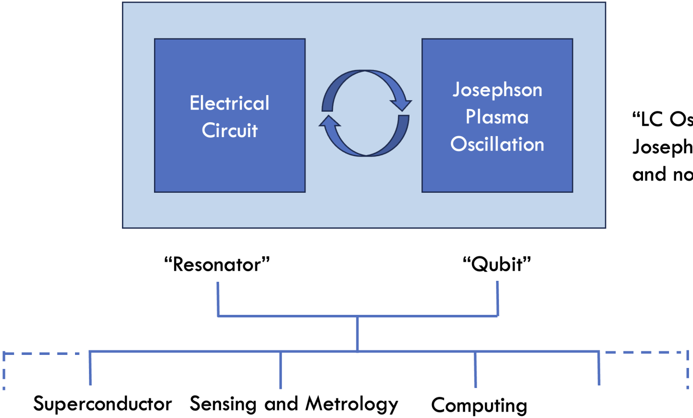
Outline of the Presentation

- Brief Introduction on Quantum Electrical Circuits
- Quantization with Symmetrization Postulate
- Quantization with Lagrange Equation
    1. Revision on Lagrangian and Hamiltonian Mechanics
    2. The Algorithm of Quantization
    3. Comparison between Symmetrization Postulate and Lagrange Equation
- Summary to the Topic: Questions List

Quantum Electrical Circuits

- The coupling between electrical oscillators and plasma oscillations in the Josephson junction



Electrical Circuit

Josephson Plasma Oscillation

"LC Oscillation" between Josephson nonlinear inductor and node capacitor

"Resonator"

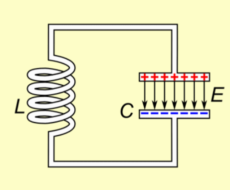
"Qubit"

Superconductor

Sensing and Metrology

Computing

Quantify LC Circuit Oscillation: Symmetrization Postulate



1. $H = \frac{1}{2}CU^2 + \frac{1}{2}LI^2 = \frac{1}{2}C\left(\frac{Q}{C}\right)^2 + \frac{1}{2}L\left(\frac{\Phi}{L}\right)^2$
2. $\alpha(t) = \frac{\Phi(t)}{A} + i\frac{Q(t)}{B} \qquad \alpha^*(t) = \frac{\Phi(t)}{A} - i\frac{Q(t)}{B}$
3. $H = \frac{AB}{\sqrt{LC}}(\alpha\alpha^* + \alpha^*\alpha) \quad AB = \frac{\hbar}{2}$
4. Quantization: introducing the canonically conjugate variables
   $\Phi \rightarrow \hat{\Phi}, Q \rightarrow \hat{Q} \quad [\hat{\Phi}, \hat{Q}] = i\hbar \qquad \alpha \rightarrow \hat{a}, \alpha^* \rightarrow \hat{a}^\dagger \quad [\hat{a}, \hat{a}^\dagger] = 1$
   $\hat{H} = \frac{\hbar}{2\sqrt{LC}}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \frac{\hbar}{\sqrt{LC}}(\hat{a}^\dagger\hat{a} + \frac{1}{2})$
5. Zero point fluctuation: $\Phi_{2pf}^2 = \langle 0|\hat{\Phi}^2|0\rangle = A^2 = \frac{\hbar}{2}\sqrt{\frac{L}{C}} \quad Q_{2pf}^2 = \langle 0|\hat{Q}^2|0\rangle = B^2 = \frac{\hbar}{2}\sqrt{\frac{C}{L}}$

Quantify LC Circuit Oscillation: Lagrangian and Hamiltonian

- Lagrange Equation
  The system can be characterized by generalized coordinates $(q_1, q_2, \dots)$ and t
  Define $\mathcal{L} = T - V \quad T = T(\dot{q}_1, \dot{q}_2, \cdots),\ V = V(q_1, q_2, \cdots)$
  Lagrange Equation: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) = \frac{\partial \mathcal{L}}{\partial q_\alpha}$
- Hamiltonian Canonical Equation
  Define generalized momentum: $p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}$
  Hamiltonian is defined: $H = \sum_\alpha p_\alpha \dot{q}_\alpha - \mathcal{L}$

Quantify LC Circuit Oscillation: Lagrangian and Hamiltonian

- Poisson Bracket
  The Poisson Bracket of $f(q_1, q_2, \dots, p_1, p_2, \dots t)$ and $g(q_1, q_2, \dots, p_1, p_2, \dots t)$
  is defined as: $\{f, g\} = \sum_i (\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i})$
  Asymmetric relation: $\{f, g\} = -\{g, f\}$
  For conjugate pair, the commutation relation is: $\{q_i, p_j\} = \delta_{ij}, \{q_i, q_j\} = 0, \{p_i, p_j\} = 0$
  Quantization is $\{q, p\} = 1 \rightarrow \frac{1}{i\hbar}[\hat{q}, \hat{p}] = 1$

Quantify LC Circuit Oscillation: Lagrangian Approach

1. $I = \dot{q}$, let q be the generalized coordinate of this system
2. Lagrangian $\mathcal{L} = T - V = T(\dot{q}) - V(q) = \frac{1}{2}L\dot{q}^2 - \frac{1}{2}\frac{q^2}{C}$
3. Lagrange Equation: $\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) = \frac{\partial \mathcal{L}}{\partial q}$, $\ddot{q} = -\frac{1}{LC}q = -\Omega^2 q$
4. The momentum conjugate of q is the generalized momentum of the system: $\frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q} = \Phi$, which is also the generalized node fluxes: $\Phi = \int^t U(\tau)d\tau$
5. Generalized coordinate q and generalized momentum Φ form the Poisson Bracket: $\{q, \Phi\} = 1$
6. Hamiltonian: $H = \dot{\Phi}q - \mathcal{L} = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}\frac{\Phi^2}{L}$
7. Quantization is $\{q, \Phi\} \rightarrow \frac{1}{i\hbar}[\hat{q}, \hat{\Phi}]$. Let $\hat{a} = \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} + i\frac{1}{\sqrt{2L\hbar\Omega}}\hat{\Phi}$, $\hat{a}^\dagger = \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} - i\frac{1}{\sqrt{2L\hbar\Omega}}\hat{\Phi}$ $[\hat{q}, \hat{\Phi}] = i\hbar \rightarrow [\hat{a}, \hat{a}^\dagger] = 1$, $\hat{H} = \hbar\Omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

Quantify LC Circuit Oscillation: Lagrangian Approach

1. Similarly, if the generalized node flux Φ is the generalized coordinate of this system
2. Lagrangian $\mathcal{L} = T - V = T(\dot{\Phi}) - V(\Phi) = \frac{1}{2}\frac{\Phi^2}{L} - \frac{1}{2}C\dot{\Phi}$
3. Lagrange Equation: $\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}) = \frac{\partial \mathcal{L}}{\partial \Phi}$
4. The momentum conjugate of Φ is the generalized momentum of the system: $\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = -C\dot{\Phi} = C\mathcal{E}_L = -CU = -q$
5. Generalized coordinate Φ and generalized momentum $-q$ fulfill the Poisson Bracket: $\{\Phi, -q\} = 1$
6. Quantization: $\{\Phi, -q\} = 1 \rightarrow [\hat{\Phi}, -\hat{q}] = i\hbar$

Quantify LC Circuit Oscillation: Comparison

Symmetrization Postulate

- Define H with the total energy of the system
- Manually introduce the canonical conjugate
- Transform variables into operators
- Quantify H into $\hat{H}$

Lagrangian Approach

- Confirm the coordinate of the system, calculate $\mathcal{L}$ for the Lagrange Equation
- Calculate the momentum conjugate of the generalized coordinate
- Deduce canonical commutation from Poisson Bracket
- Quantify H into $\hat{H}$

Summary to the Question List

1. What is the algorithm for the quantization procedure using Lagrangian mechanics?
2. What is the main benefit of the Lagrangian approach compared to the approach utilizing symmetrization postulate?
    - Deduce canonical commutation relation directly from Hamiltonian Mechanics, without doing manually hypothesis, making it easier and clearer to figure out the canonical commutator
3. How does one identify the variables that should become operators with the canonical commutator?
    - Generalized coordinate and its conjugate momentum should be the canonical commutators
4. What is the classical analog of the canonical commutation relation?
    - Poisson Bracket, and the quantization is $\{q, p\} \rightarrow \frac{1}{i\hbar}[\hat{q}, \hat{p}]$
5. What are the generalized node fluxes? How are they defined? How do they correspond to other possible variables (charges, currents, voltages, magnetic fluxes through inductors)?
    - Generalized node flux is particularly defined in electrical circuit oscillators, as the integral of node voltage: $\Phi = \int^{t} U(\tau) d\tau = L\dot{q} = LI$, which is the magnetic fluxes through inductors of the LC circuit